tion (4. 1) changes them proportionally. Moreover when the perturbation deviates from its worst mode, the fictitious initial time t^* increases and this leads, by virtue of (4. 1), to decrease in the values of α' and k'. Thus the network of parameter α' (or k') represents a decomposition of the segment $[0, (T - t_0^*)\alpha]$, where t_0^* is a fictitious initial time corresponding to the initial conditions of the problem.

In this manner, the initial correction problem which has, in the case of a constant intensity, eight parameters x_0, y_0, T, α, k, p, Q and n is reduced, by means of transformation (4. 1), to a problem with four parameters α', k', Q' and n. Thus a synthesis of a correction for a real object requires only n one-parameter (α' or k') relations $t_1'(Q')$.

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QUASI-NORMAL AND NORMAL OSCILLATIONS IN CONSERVATIVE SYSTEMS

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Particular kinds of periodic solutions in unison — quasi-normal oscillations similar to those of an oscillator — are separated in conservative multidimensional systems. A new definition of normal oscillations, more precise than known ones is proposed. It is applicable to a wider class of nonlinear systems. A method of approximate determination of quasi-normal oscillations for a particular kind of nonlinear systems is described and some examples are presented.

In [1, 2] the supposition was made that singular analogs of characteristic solutions, often called normal oscillations, can exist in the class of nonlinear conservative systems of the form $x^{\cdot\cdot}_i = \partial U / \partial x_i$, $U(0) = 0$, $U(-x) = U(x)$ and $x = \{x_1, x_2, \ldots, x_n\}$. It was assumed that normal oscillations are determined by the following characteristic properties: oscillation frequencies of all coordinates are equal, all coordinates attain their maximum deflection and vanish simultaneously, and the displacement of coordinates at any instant of time is a single-valued function of one of these.

From the physical point of view the above definition of normal oscillations has the following shortcomings: the characteristic properties of normal oscillations are noninvariant under the change of the coordinate system, are interdependent, comprise a narrow class of nonlinear systems, and do not permit the formulation of the problem of determining normal oscillations.

In this paper the concept of normal oscillations of nonlinear systems is extended, a new definition of strictly normal oscillation is presented, and the algorithm for determining quasi-normal oscillations is formulated for the class of strongly nonlinear systems which is of practical interest.

1. Quasi-normal oscillations. Let us consider a conservative system with Lagrangian $L = L(x, x^{\cdot})$, $x = \{x_1, \ldots, x_n\}$ that is continuous over the set of arguments and has continuous partial derivatives up to the third order. We assume that

$$\det\left(\frac{\partial p_i}{\partial x_j^{\cdot}}\right) \neq 0 \quad (p_i = L_{x_i^{\cdot}}), \quad i, j = 1, \ldots, n \tag{1.1}$$

Further restrictions that can be conveniently formulated for the Hamiltionian $H = H(x, p)$ are given below.

The equations of motion

$$p_i^{\cdot} = L_{x_i} \tag{1.2}$$

are extrema of the functional

$$\int\limits_{x(t_0)}^{x(t_1)} L dt \tag{1.3}$$

whose variation consists of two parts: the integral extended over the given interval and of the boundary term

$$\delta \int\limits_{x(t_0)}^{x(t_1)} L dt = \int\limits_{x(t_0)}^{x(t_1)} \sum_{i=1}^{n} (p_i^{\cdot} - L_{x_i}) h_i(t)\, dt + \sum_{i=1}^{n} p_i \delta x_i \Big|_{t_0}^{t_1}$$

In problems of the theory of oscillations it is usually assumed that variation is carried out for fixed boundary conditions and $\delta x_i = 0$. However the boundary conditions that determine periodic solutions, in particular the normal oscillations are not *a priori* known. The separation of normal oscillations is based in [1] on the adoption of certain properties of trajectories, as a whole, that are specific for normal oscillations of linear systems. Another possibility of extending this concept consists of an appropriate selection of limits of integration in (1.3).

Definition 1. We call quasi-normal oscillations the solutions of Lagrange equations that satisfy the reasonable boundary conditions

$$p_i \Big|_{t_0}^{t_1} = 0$$

The simultaneous vanishing of generalized momenta in two noncoincident instants of time are the mathematical expression of the intuitive idea of motion in unison. In linear systems quasi-normal oscillations coincide with normal ones, if all natural frequencies are incommensurable, otherwise quasi-normal oscillations constitute a parameteric set which comprises the strictly normal single-frequency oscillations.

Let us establish some of the properties of quasi-normal oscillations.

The energy integral $H(x, p) = h$ determines surface m in the phase space $\Phi \ni (x, p)$. We denote in the space of configurations $E \ni (x, 0)$ the surface defined by $H(x, 0) = h$ by M. We assume that sets m^* and M^* bounded by surfaces m and M are compact and connecting. The following statement, whose validity is based on simple geometrical considerations, is used below.

Statement 1. If the orthogonal projection of m^* on E belongs to M^*, then any trajectory of the dynamic system belongs to M^*.

Here and subsequently the trajectory is understood to be the projection of the phase trajectory onto the region of the space of E-configurations.

Conservative systems whose Hamiltonian satisfies the above requirements are widespread. As a possible example we adduce a system with the Hamiltonian $H=U(x)+T(x, p)$, $T>0$ for $p\neq 0$ and $T=0$ for $p=0$. Dynamic systems of this kind are an extension of systems considered earlier in [1].

The condition $p_i(t_0)=0$ is equivalent to the following: at some instant of time t_0 the extremum of functional (1.3) intersects surface M. At that intersection point the condition of transversality must be satisfied. We formulate this property of trajectories in the form of a statement.

Statement 2. The normalized vector of generalized momenta is orthogonal to surface M, i.e.

$$\lim \frac{p_i}{p_1} = \lim \frac{H_{x_i}}{H_{x_1}}, \quad p_i \to 0 \tag{1.4}$$

Formula (1.4) can be established without resorting to the formalism of variations by extending it to systems of the more general form

$$p_i^{\cdot}(x, x^{\cdot}, t) = L_{x_i}(x, x^{\cdot}, t) + Q_i(x, x^{\cdot}, t)$$

Applying the l'Hospital's rule, we write

$$\lim \frac{p_i}{p_1} = \lim \frac{p_i^{\cdot}}{p_1^{\cdot}} = \lim \frac{L_{x_i}+Q_i}{L_{x_1}+Q_1}, \quad p_i \to 0 \tag{1.5}$$

For $Q_i=0$ formula (1.5) reduces to (1.4). This becomes clear, if we take into account that by virtue of (1.1) the relationships $p_i = L_{x_i^{\cdot}}$ are solvable for $x_i^{\cdot}$ and $L = \Sigma p_i x_i^{\cdot} - H$, and that by exchanging the sequence of operations of differentiation and of passing to limit we obtain the equality of the right-hand parts of formulas (1.4) and (1.5).

Corollary. The dynamic system trajectories do not intersect surface M. This corollary follows from Statement 2 and conditions of smoothness of L which ensure the uniqueness of grad H.

Statement 3. Quasi-normal oscillations are periodic solutions whose trajectories intersect M at two different points.

Let us assume the opposite. Let t_0 and t_1 $(t_0<t_1)$ be consecutive instants of time at which the quasi-normal trajectory that entirely belongs to region M^* (Statement 1) intersects its boundary M. By assumption t_1 is the salient point of the quasi-normal trajectory, which contradicts the corollary.

By analogy to linear systems the motions of a multi-dimensional conservative nonlinear system along each quasi-normal trajectory can be interpreted as oscillations of an oscillator.

Let us assume for simplicity that E is Euclidean, i.e. that $p=x^{\cdot}$. We select the direction of circumvention and determine the metric along the curve in E as the distance of the running point from some point (the reference point) of the curve.

Statement 4. Only a quasi-normal trajectory is homeomorphic to a closed segment in the topology induced by the metric on the curve.

The homeomorphism of the quasi-normal trajectory to the segment is evident. The trajectory of a periodic nonquasi-normal solution (because the equalities $x_i^{\cdot}=0$ are not satisfied simultaneously) must be a closed curve in E that is not homeomorphic to the segment; the trajectory of a nonperiodic solution is noncompact, and also nonhomeomorphic to the closed segment.

It can be intuitively appreciated that oscillations of an oscillator differ from other motions, for instance of rotation in that the oscillating motion is periodic and that at peak points the velocity is zero. These properties are equivalent to the single condition that the trajectory of the oscillating mode is homeomorphic to the closed segment. Hence only quasi-normal oscillations in multi-dimensional conservative systems can be interpreted as oscillations of an oscillator. Moreover, it is not possible by a transformation of coordinates to separate in nonlinear systems one or several equations (related to motions along quasi-normal trajectories) unconnected with the remaining ones.

Let us illustrate Statement 4 by a simple example. The derived formulas will be used subsequently. It can be readily demonstrated that when the ratios of natural frequencies $\omega_i / \omega_1 = l_i$ of a linear system are integers, the motion trajectory is determined by formula

$$x_j = a_j \left[T_{l_j}\left(\frac{x_1}{a_1}\right) \cos\varphi_j + \sqrt{1 - T^2_{l_j}\left(\frac{x_1}{a_1}\right)} \sin\varphi_j \right], \qquad j = 2, 3, \ldots, n \quad (1.6)$$

where T_{l_j} is a Chebyshev polynomial of the first kind, a_j are amplitudes, and φ_j are phases of the i-th normal oscillation. Substituting (1.6) into the energy integral

$$\sum_{j=1}^{n} (x_j'^2 + l_j^2 \omega_1^2 x_j^2) = 2h$$

we obtain

$$x_1'^2 \left\{1 + \sum_{j=2}^{n} \left[T'_{l_j} \cos\varphi_j + (T_{l_j} T'_{l_j} \sin\varphi_j) / \sqrt{1 - T^2_{l_j}}\right]^2 \right\} \frac{a_j^2}{a_1^2} + Q(x_1) = 2h \quad (1.7)$$

Formula (1.7) can be considered as the energy integral of the equivalent nonlinear oscillator with a variable mass. For $x_1 \to a_i$ the "mass" of the equivalent oscillator is bounded only for quasi-normal oscillation (i.e. for $\varphi_i = \pm k\pi,\ k = 1, 2, \ldots$).

Methods of group theory can be used for simplifying the equations of motion and determining quasi-normal oscillations. In this an important part is played by manifolds that are invariant with respect to finite and continuous groups admitted by the equations of motion.

Let a finite group of transformations $G \supset g_i$ be specified. We denote by S_{g_i} the representation of G in E ($S_{g_i}(x) = x^*,\ x,\ x^* \in E$). The invariant manifold of group G in E is determined by equations

$$\operatorname{inv} E_G\colon\ S_{g_i}(x) = x, \quad g_i \subset G$$

S t a t e m e n t 5. If L is invariant with respect to a finite group of coordinate transformations, then the invariant manifold of the group is the integral manifold of system (1.2).

P r o o f. Let us assume for simplicity that function S_{g_i} is continuously differentiable and that the invariant manifold has the common rank equal N.

Let us show that the dimension of the Lagrange system of equations projected onto $\operatorname{inv} E_G$ is N. We carry out the nondegenerate transformation of coordinates $\{x_1, \ldots, x_n\} \to \{U_1, \ldots, U_N, V_{N+1}, \ldots, V_n\}$ so that the coordinate curves U_i belong to $\operatorname{inv} E_G$, and V_i intersect $\operatorname{inv} E_G$ in not more than at one point. The nondegenerate transformations S_{g_i} are expressed in new coordinates by

$$S^{(1)}_{g_i}(U) = U^*, \quad S^{(2)}_{g_i}(U, V) = V^*, \quad U, V, U^*, V^* \in E \quad (1.8)$$

Differentiation of these equalities yields formulas for the transformation of derivatives

$$\sum_{j=1}^{N} \frac{\partial S_{g_i}^{(1)}}{\partial U_j} U_j^{\cdot} = U^{\cdot *}, \quad \sum_{j=1}^{N} \frac{\partial S_{g_i}^{(2)}}{\partial U_j} U_j^{\cdot} + \sum_{j=N+1}^{n} \frac{\partial S_{g_i}^{(2)}}{\partial V_j} V_j^{\cdot} = V^{\cdot *} \qquad (1.9)$$

The definition of the invariant manifold of the group implies that

$$S_{g_i}^{(2)} (U, 0) \equiv 0 \qquad (1.10)$$

The invariance condition is of the form

$$L (U, V, U^{\cdot}, V^{\cdot}) = L (U^*, V^*, U^{\cdot *}, V^{\cdot *})$$

After computation of derivatives in both parts of this identity

$$\frac{\partial L}{\partial V} = \frac{\partial L}{\partial V^*} \frac{\partial V^*}{\partial V} + \frac{\partial L}{\partial V^{\cdot *}} \frac{\partial V^{\cdot *}}{\partial V}, \qquad \frac{\partial L}{\partial V^{\cdot}} = \frac{\partial L}{\partial V^{\cdot *}} \frac{\partial V^{\cdot *}}{\partial V^{\cdot}}$$

we obtain in accordance with formulas (1.8) – (1.10) that $\partial L / \partial V = 0$, $\partial L / \partial V^{\cdot} = 0$ for $V = 0$ and $V^{\cdot} = 0$.

Statement 5 , an analog of the Noether theorem, remains valid when the invariant manifold degenerates into a point which, for instance, occurs with coordinate inversion.

To find the most complete set of integral manifolds it is necessary to determine the manifolds that are invariant with respect to all subgroups of group G.

Statement 6. If L is invariant with respect to group G, then the set of quasi-normal oscillations is also invariant with respect to G.

This is so because the invariance of L implies the invariance of the equations of motion and of boundary conditions $(p_i = 0)$ which determine quasi-normal oscillations.

Corollary. If L is invariant under coordinate inversion, the set of quasi-normal oscillations is also invariant under inversion.

Nevertheless separate quasi-normal and, as noted below, also normal trajectories may not be invariant under inversion and may not pass through the inversion center. An example of this appears in Sect. 3. If the quasi-normal trajectory does not pass through the inversion center, then one more quasi-normal trajectory conjugate of the first with respect to the inversion center, and, consequently, of the same length can be found.

The presented definition of the quasi-normal oscillations makes it possible to use numerical methods of solution of nonlinear boundary value problems for finding such oscillation modes. Exact solutions which may be obtained for systems admitting a fairly wide group – finite or continuous – of transformations, can be taken as the initial approximation.

2. Normal oscillations. If quasi-normal oscillations form a parametric set in nonlinear, as well as in linear systems, it is expedient to separate the strictly normal oscillations from the former. By analogy with linear systems a rectilinear trajectory can be taken as the characteristic sign of normal oscillations in nonlinear systems.

Condition (1.5) must be satisfied at all points of the rectilinear trajectory along which the representing point is moving. This can be used for finding rectilinear normal trajectories. Substituting $x_i = \alpha_i x_1 + \beta_i$ and $x_i^{\cdot} = \alpha_i x_1^{\cdot}$ into (1.5) we obtain a system of equations in α_i and β_i which must be identically satisfied with respect to x_1 and $x_1^{\cdot}$. If L is analytic over the set of arguments, functions x_1 and $x_1^{\cdot}$ can be eliminated by equating to zero the coefficients at all powers of x_1 and $x_1^{\cdot}$. The system of equations derived in this manner can only be consistent for some particular L and Q. However in certain problems interesting from the point of view of analysis, normal trajectories are rectilinear and it is possible to determine in this manner the modal constants α_i and β_r.

Since the application of the test of the trajectory rectilinearity is limited, the separation of normal solutions necessitates the use of more fundamental properties. It can be shown that normal trajectories of linear systems have a local minimum length in the set of trajectories that intersect M. We shall use the similar property for determining normal oscillations in nonlinear systems.

Definition 2. We call a trajectory normal, if it has a minimum local length in the set of trajectories that intersect surface M.

The differential of the trajectory arc length is taken in the form $ds = \sqrt{\Sigma p_i x_i^{\cdot}} dt$. Normal oscillations defined in this manner constitute a subclass of quasi-normal (oscillations), since among solutions that intersect H only quasi-normal trajectories are of finite length. Hence a normal trajectory coincides with one of the quasi-normal.

We assume that E is Euclidean and shall show that rectilinear trajectories (when they exist) are normal. For this we consider the functional

$$s = \int \sqrt{\sum x_i^{\cdot 2}}\, dt \qquad (2.1)$$

on the set of continuous curves with piecewise continuous derivative, whose ends lie on surface M. Extrema of this functional are straight lines that intersect orthogonally surface M. Rectilinear trajectories orthogonal to M are found between these. Thus, solutions containing rectilinear trajectories yield the extremum of functional (2.1) in the set of curves containing the set of quasi-normal trajectories. They consequently also yield the extremum of functional (2.1) in the set of quasi-normal trajectories. It is obvious that that extremum is, in fact, a minimum.

Although the quasi-normal and normal oscillations were not explicitly determined, we shall prove the existence of related solutions. The proof is based on the theorem about the minimum of the lower semicontinuous functional.

Since all trajectories belong to the compact region M^*, we consider functional (1.3) on the set of continuous rectifiable curves with piecewise continuous derivative, which belong to M^*, and whose ends lie on M. It is sufficient [3 — 5] to show that the functional (1.3) attains its absolute minimum in the described set of curves. The latter according to [3 — 5] takes place if on the curves that determine functional (1.3) L is nonnegative and $\det |\partial p_i / \partial x_j^{\cdot}| > 0$. In conventional problems of mechanics L, as the difference between two bounded quantities of kinetic and potential energy, is a bounded quantity. Since the equations of motion are not affected by the addition to L of some constant quantity, hence, without loss of generality, we assume that $L > 0$.

Statement 7. If p_i are continuous and twice continuously differentiable, $L > 0$ and $\det |\partial p_i / \partial x_j^{\cdot}| > 0$, functional (1.3) attains in the described set of curves its absolute minimum in the quasi-normal trajectory.

Let us assume that the quasi-normal solutions form a parameteric set, that function

$$\sqrt{\sum p_i x_i^{\cdot}} = Q$$

is nonnegative, and $\det |\partial^2 Q / \partial x_i^{\cdot} \partial x_j^{\cdot}| > 0$. Then at least one normal trajectory of minimum length exists among quasi-normal trajectories of the set.

3. Systems with a small parameter. Let us describe the algorithm for determining normal oscillations of the class of nonlinear systems whose equations of motion are of the form

$$Q(x^{\cdot\cdot}, x^{\cdot}, x) + \varepsilon q(x^{\cdot\cdot}, x^{\cdot}, x, y^{\cdot\cdot}, y^{\cdot}, y, \varepsilon) = 0 \qquad (3.1)$$
$$F(x^{\cdot\cdot}, x^{\cdot}, x, y^{\cdot\cdot}, y^{\cdot}, y, \varepsilon) = 0$$

where

$$Q = \{Q_1, \ldots, Q_n\}, \quad q = \{q_1, \ldots, q_n\}, \quad x = \{x_1, \ldots, x_n\}, \quad x^{\cdot} = \{x_1^{\cdot}, \ldots, x_n^{\cdot}\}$$
$$F = \{F_1, \ldots, F_m\}, \quad y = \{y_1, \ldots, y_m\}, \quad y^{\cdot} = \{y_1^{\cdot}, \ldots, y_m^{\cdot}\}$$

We assume that for $\varepsilon = 0$, $y = 0$ and $F = 0$, and that the equations

$$Q(x^{\cdot\cdot}, x^{\cdot}, x) = 0 \qquad (3.2)$$

have a set of quasi-normal solutions which depend on the arbitrary constants

$$x_i^{(0)} = x_i^{(0)}(t, c), \quad y = 0, \quad c = \{c_1, \ldots, c_n\}$$

We have to determine constants c for which the quasi-normal oscillations of the input and simplified equations (3.1) and (3.2), respectively, are asymptotically close.

The method of small parameter applied to a similar problem involves cumbersome conditions of solution periodicity with periodic coefficients and periodic right-hand part [6]. We formulate below a simpler method, based on Statement 2, for determining constants c in the case of quasi-normal oscillations.

Taking into account that y and $y^{\cdot} = O(\varepsilon)$ do not appear either in the sought solution or in the generating equation (3.2), we conclude that the neglect of functions y and $y^{\cdot}$ in (3.1) leads to an error of order ε^2. Hence we select c so that for quasi-normal solutions of zero approximation the boundary condition (1.4) or (1.5) are satisfied with an accuracy to within terms of order ε^2

$$\lim p_i / p_1 = \lim H_{x_i}^* / H_{x_1}^*, \quad p_i \to 0 \quad (i = 2, 3, \ldots, n) \qquad (3.3)$$
$$H = H_0 + \varepsilon H_1 + \ldots, \quad H^* = H_0 + \varepsilon H_1$$

In function H^* vector y of order ε^2 is equated to zero. Condition (3.3) makes it possible to form a closed system of transcendental equations that associate constants c.

Statement 8. Condition (3.3) is sufficient for (the existence of) asymptotic closeness of quasi-normal trajectories in a finite time interval.

Condition (3.3) ensures an asymptotic closeness of quasi-normal trajectories of Eqs. (3.1) and (3.2) at a point of surface M. In the compact region M^* systems (3.1) and (3.2) are δ-close to each other in the sense of [7]. The asymptotic closeness of trajectories at the origin implies asymptotic closeness of these in a finite interval of time that contains that point [7]. Hence (3.3) is a sufficient condition of asymptotic closeness of quasi-normal trajectories of the input and the simplified systems in a finite time interval.

In a number of problems the generating solution is a set of straight lines $x_i = c_i x_1$. In that particular case formula (3.3) yields the exact solution.

We present some examples which illustrate the effectiveness of the described method and the characteristic properties of quasi-normal oscillations of nonlinear systems. Let us write the equations of motion as

$$x_i^{\cdot\cdot} + \omega_i^2 x_i + \varepsilon f_i(x) = 0, \qquad x_k^{\cdot\cdot} + \omega_k^2 x_k + \varepsilon f_k(x) = 0 \qquad (3.4)$$
$$x = \{x_1, \ldots, x_n\}, \qquad i = 1, 2, \ldots, m, \qquad k = m + 1, \ldots, n$$

We assume that the ratios $\omega_i / \omega_1 = l_i$ are integers and that ω and ω_k are incommensurable. Let us determine the quasi-normal trajectories of system (3.4) that are asymptotically close to quasi-normal trajectories $x_i = a_i T_{l_i}(x_1 / a_1)$ of the linearized system. Taking into consideration that $x_i^{\cdot} / x_1^{\cdot} = dx_i / dx_1$, we represent (3.3) as

$$a_i T'_{l_i}\left(\frac{a_i}{a_1}\right) = \frac{\omega_i^2 a_i + \varepsilon f_i(a)}{\omega_1^2 a_1 + \varepsilon f_1(a)} \qquad \text{for} \quad U = h \tag{3.5}$$

where U is a potential function.

Example 1. We write the equations of motion in the form

$$x_1^{\cdot\cdot} + \omega^2 x_1 + \varepsilon x_1 x_2^2 = 0, \qquad x_2^{\cdot\cdot} + 4\omega^2 x_2 + \varepsilon x_1^2 x_2 = 0$$

Equation (3.5) has in this case four solutions

$$x_1 = 0, \qquad x_2 = 0; \qquad x_1 = 2x_2, \qquad x_1 = -2x_2,$$

which with an accuracy to within terms of order ε^2 are the same as the loci of convergence of quasi-normal trajectories. The first pair of orthogonal straight lines coincides with quasi-normal trajectories, while the second determines approximately initial conditions of the other two quasi-normal trajectories which for not very great h are close to parabolic trajectories of the linearized system.

Results of numerical analysis carried out on a computer using the Runge-Kutta algorithm are shown in Fig. 1 for $\varepsilon = 1$ $\omega = 1$ and $h = 8$. Trajectories of quasi-normal

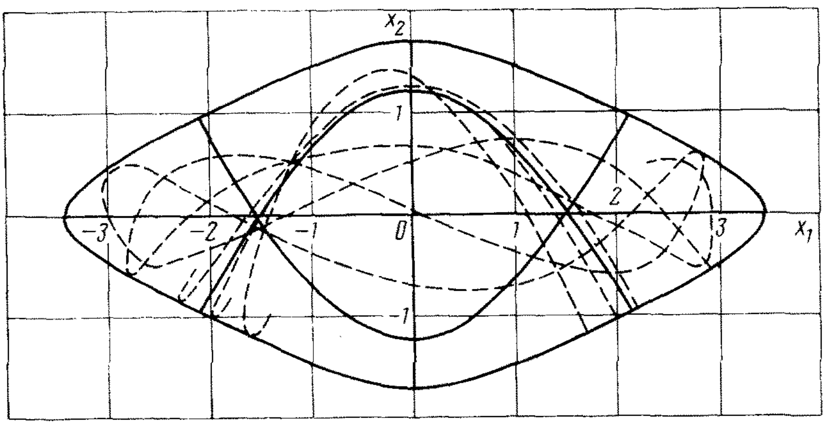

Fig. 1

oscillations are shown by solid lines, while the dash lines show the trajectories of nonperiodic solutions that intersect the equipotential curve.

The trajectory emanating from an arbitrary point of the equipotential curve

$$\tfrac{1}{2}(x_1^2 + 4x_2^2) + \tfrac{1}{2} x_1^2 x_2^2 = 6$$

is not a periodic curve. Trajectories of equipotential oscillations are clearly distinguishable among the nonperiodic solutions whose trajectories are very complicated.

The error of analytic formulas defining initial conditions of curvilinear quasi-normal trajectories is approximately 0.7% for $h = 0.6$ and 4% for $h = 6$.

Numerical analysis of the considered oscillating system shows that the set of quasi-normal oscillations is discrete. For such systems (see Sect. 2) the quasi-normal and normal oscillations coincide. We have thus determined that in a system whose Lagrangian is invariant under coordinate inversion there are four normal trajectories two of which are rectilinear and two curvilinear which do not pass through the inversion center.

Certain classes of nonlinear systems whose potential is a homogeneous function of coordinates, are distinguished by the number of trajectories being greater than the number of degrees of freedom [1]. However for the considered systems whose potential is not a homogeneous function of coordinates and contains a quadratic component, this phenomenon is apparently noted for the first time. It occurs when all normal trajectories are rectilinear.

Example 2. We write the equations of motion thus:

$$x_1^{\cdot\cdot} = -\omega^2 x_1 - b x_1 x_2^2 - e x_1^3, \qquad x_2^{\cdot\cdot} = -\omega^2 x_2 - b x_1^2 x_2 - d x_2^3$$

From (3.5) we obtain for the four normal trajectories the equations

$$x_1 = 0, \quad x_2 = 0, \quad x_1 = \pm \lambda x_2, \quad \lambda = \sqrt{b-e} \,/\, \sqrt{b-d}$$

Normal oscillation trajectories are shown in Fig. 2 for $\omega = \sqrt{2}$, $b = 4$ and $d = e = 8$ by solid straight lines, and the trajectories of nonperiodic solutions that intersect the equipotential curve are represented by dash lines.

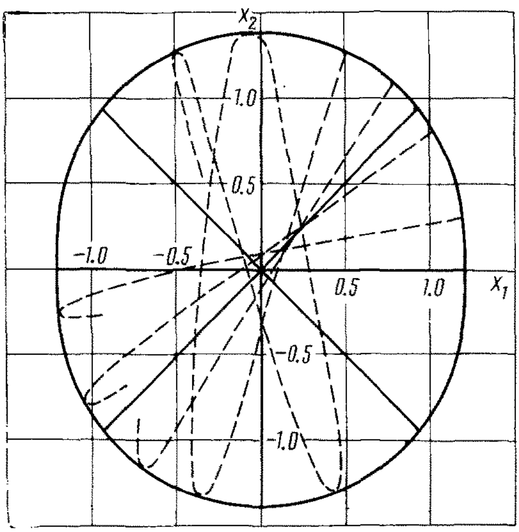

Fig. 2

This example provides a pictorial geometric interpretation. The equipotential curves of the linearized system constitute a set of concentric circles whose any diameter coincides with a normal trajectory. The superposition of any arbitrarily small perturbation destroys the circular symmetry and leads to the appearance of several predominant directions that coincide with the directions of normal trajectories. Condition (3.5) makes it possible to obtain these solutions in an explicit form.

Formula (3.5) makes possible the separation of the basic component in the approximate representation of quasi-normal trajectories. The solution can be refined by various means. For this it is expedient to use the Galerkin method. It is convenient to select the coordinate functions in the form of polynomials of space coordinates. Efficiency of the Galerkin method is explained by that in zero approximation the shape of oscillations can usually be determined fairly accurately. The unknown weighting coefficients at coordinate functions are in this case small, which makes it possible to linearize in the first approximation the system of transcendental equations that link these.

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DYNAMICS OF A GRAVITATING GASEOUS ELLIPSOID

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Dynamics of adiabatic motions of a gravitating perfect gas of constant density filling a certain ellipsoid is considered in the case when velocities are linear functions of coordinates. It is shown that for an adiabatic exponent $\gamma < 4/3$ the spherically symmetric compression of gas into a point is an unstable process. A reasonable approximation of the oscillating gas motion under strong compression for considerable negative gas energy is indicated. Oscillating mode of expansion of a rotating gaseous ellipsoid in vacuum, which obtains also in the absence of gravitational interaction between gas particles, is determined.